System of differential equations

Given the following system:

$$\begin{cases} x'(t) = 2x(t) - 2y(t) + t + 1, \\ y'(t) = 5x(t) - 4y(t) + 2 + \sin(t), \end{cases}$$

a) Solve the associated homogeneous system.

b) Analyze stability.

c) Outline, without solving, how the system would look with its non-homogeneous solution.

Solution

(a)

The associated homogeneous system is:

$$\begin{cases} x'(t) = 2x(t) - 2y(t), \\ y'(t) = 5x(t) - 4y(t). \end{cases}$$

We can write this in matrix form:

$$\mathbf{X}'(t) = A\mathbf{X}(t),$$

where:

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -2 \\ 5 & -4 \end{pmatrix}.$$

We calculate the eigenvalues λ by solving the characteristic equation:

$$\det(A - \lambda I) = 0.$$

$$\det\left(\begin{pmatrix} 2-\lambda & -2 \\ 5 & -4-\lambda \end{pmatrix}\right) = (2-\lambda)(-4-\lambda) - (-2)(5).$$

$$(2-\lambda)(-4-\lambda) + 10 = \lambda^2 + 2\lambda + 2.$$

Solving:

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i.$$

The eigenvalues are:

$$\lambda_1 = -1 + i, \quad \lambda_2 = -1 - i.$$

We find the eigenvectors $\mathbf{v}$ that satisfy:

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

For $\lambda = -1 + i$, we have:

$$A - \lambda I = \begin{pmatrix} 2-(-1+i) & -2 \\ 5 & -4-(-1+i) \end{pmatrix} = \begin{pmatrix} 3-i & -2 \\ 5 & -3-i \end{pmatrix}.$$

Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. The system is:

$$\begin{cases} (3-i)v_1 - 2v_2 = 0, \\ 5v_1 + (-3-i)v_2 = 0. \end{cases}$$

From the first equation:

$$(3-i)v_1 = 2v_2 \implies v_2 = \frac{(3-i)}{2}v_1.$$

The eigenvector is:

$$\mathbf{v}_1 = v_1 \begin{pmatrix} 1 \\ \frac{3-i}{2} \end{pmatrix}.$$

For $\lambda_2 = -1 - i$, the eigenvector is the complex conjugate of $\mathbf{v}_1$:

$$\mathbf{v}_2 = v_1^* \begin{pmatrix} 1 \\ \frac{3+i}{2} \end{pmatrix}.$$

The general solution of the homogeneous system is:

$$\mathbf{X}_h(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Using the identity:

$$e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t),$$

with $\lambda_1 = -1 + i$, so $\alpha = -1$ and $\beta = 1$, the solution becomes:

$$\mathbf{X}_h(t) = e^{-t}\left[C_1 e^{it}\mathbf{v}_1 + C_2 e^{-it}\mathbf{v}_2\right].$$

(b)

The stability of the system depends on the eigenvalues of the matrix A:

$$\lambda = -1 \pm i.$$

Since the real part of both eigenvalues is negative (-1), the system is asymptotically stable.

(c)

For the non-homogeneous system:

$$\mathbf{f}(t) = \begin{pmatrix} t+1 \\ 2+\sin(t) \end{pmatrix},$$

we propose a particular solution of the form:

$$\mathbf{X}_p(t) = \begin{pmatrix} at + b + A\cos(t) + B\sin(t) \\ ct + d + C\cos(t) + D\sin(t) \end{pmatrix},$$

where a, b, c, d, A, B, C, D are constants to be determined by substituting into the equation.